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(4,2)-CHAIN HOMOTOPY

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We consider (4,2)-chain homotopy for (4,2)-chain maps between (4,2)-chain complexes (weak or strong), and prove that if f and g are (4,2)-chain homotopic, then they induce the same homomorphisms on the (4,2)-homology groups for the correspondent (4,2)-chain complexes.

Key words: commutative (4,2)-group; weak (4,2)-chain complex; strong (4,2)-chain complex; (4,2)-chain map; (4,2)-chain homotopy

INTRODUCTION

The notions of (4,2)-chain complexes and (4,2)-chain homology groups were introduced and examined in [5]. In this paper we consider a notion of a (4,2)-chain homotopy, analogouos to the usual notion of a chain homotopy for chain complexes. Although the introduced notion of a (4,2)-chain homotopy, in general, does not behave in the same way as the usual chain homotopy (for example, the relation among (4,2)-chain maps defined by (4,2)-chain homotopies is not an equivalence), it produces the same results on the (4,2)-homology groups.

For the usual notions about chain complexes of Abelian groups, chain homotopy and homology groups we refer to [4]. We recall the basic notions and properties about (4,2)-groups and (4,2)-chain complexes from [1], [2], [3] and [5].

1º A (4,2)-semigroup is a pair (G,[ ]), where G is a nonempty set and [ ]: $G^4 \to G^2$ is a (4,2)-operation, such that for any $x,y,z,t,u,v \in G$,

$$[[xyzt]uv] = [x[yztu]v] = [xy[ztuv]],$$

i.e. [ ] is (4,2)-associative.

Since the (4,2)-operation is associative, we use the notation [xyztuv] for [[xyzt]uv].

For (4,2)-semigroups (G,[ ]), (G',[ ]'), a (4,2)-homomorphism is a map f: $G \to G'$ such that [f(x)f(y)f(z)f(t)] = (f(u),f(v)), where (u,v) = [xyzt] for any $x,y,z,t \in G$.

Any (4,2)-semigroup (G,[ ]) induces a semigroup $(G^2,\circ)$, where "$\circ$" is the binary operation on G^2 defined by:

$$(x,y)\circ(u,v)=[xyuv],$$

for any $(x,y),(u,v)\in G^2$.

We say that a (4,2)-semigroup (G,[ ]) is a commutative (4,2)-group if $(G^2,\circ)$ is a commutative group.

2º Let (G,[ ]), be a commutative (4,2)-group. Then there is $0\in G$ and for each $x\in G$, there is a unique element $-x\in G$, such that for any $x,y,z,t\in G$:

(a) [xyzt] = [zyxt] = [xtzy] = [ztxy];
(b) if [xyzt] = (u,v), then [yxzt] = (v,u);
(c) [00xy] = (x,y) and [xx(-x)(-x)] = (0,0);
(d) if [xxyy] = (u,v), then u = v;
(e) if [x(–x)y(–y)] = (u,v), then v = –u;
(f) the neutral element in $(G^2,\circ)$ is (0,0); and
(g) the inverse element for $(x,y)\in(G^2,\circ)$ is the element $(x,y)^{-1}$ = [yx(–x)(–x)(–y)(–y)].

3º A subset H of G, for a given commutative (4,2)-group (G,[ ]), is a (4,2)-subgroup, if $u,v\in H$, for any $x,y,z,t\in H$ with [xyzt] = (u,v).

For a (4,2)-subgroup (H,[ ]) of a commutative (4,2)-group (G,[ ]), in general, there is no way of defining a (4,2)-factor group, but for normal (4,2)-subgoups (4,2)-factor groups are defined.

4° A (4,2)-subgroup (H,[ ]) a of a commutative (4,2)-group (G,[ ]) is said to be normal, if

$$[x_1x_2H^2] = [y_1y_2H^2] \Leftrightarrow [x_jx_jH^2] = [y_jy_jH^2],$$

for any $x_1,x_2,y_1,y_2\in G$, and j=1,2, where

$$[xyH^2] = \{ [xyuv] \mid u,v\in H \}.$$

If (H,[ ]) is a normal (4,2)-subgroup of a commutative (4,2)-group (G,[ ]), the (4,2)-factor group (G/H,[ ]) is defined by: $G/H = \{ x^{\sim} \mid x\in G\}$, where ~ is the equivalence relation on G defined by

$$x\sim y \Leftrightarrow [xxH^2] = [yyH^2] \text{ i.e. } x - y\in H,$$

and $[x^{\sim}y^{\sim}z^{\sim}t^{\sim}] = (u^{\sim},v^{\sim})$ for $[xyzt] = (u,v)$.

5° The commutative (4,2)-groups and (4,2)-homomorphisms, form the category (4,2)-*Ab*.

Three functors, denoted by Φ_2, Φ_+ and Φ_* from the category (4,2)-*Ab* to the category *Ab* of commutative groups are defined as follows.

For a commutative (4,2)-group $\underline{G}$ = (G,[ ]):

(1) $\Phi_2(\underline{G})$ is the group $(G^2,\circ)$, defined in **1°**;

(2) $\Phi_+(\underline{G})$ = (G,+), where x+y =u if and only if [xxyy] = (u,u); and

(3) $\Phi_*(\underline{G})$ = (G,$*$), where $x * y = u$ if and only if $[x(-x)y(-y)] = (u,-u)$.

If f:G → G' is a (4,2)-homomorphism, then, $\Phi_+(f) = \Phi_*(f) = f$ and $\Phi_2(f):G^2 \to (G')^2$ is defined by $\Phi_2(f)(x,y) = (f(x),f(y))$.

6° By analogy with the notion of a chain complex of Abelian groups, two types of (4,2)-chain complexes of commutative (4,2)-groups, introduced in [5], are defined as follows.

A weak (4,2)-chain complex, denoted by w(K,∂) , is a sequence

$$\ldots\leftarrow(K_{n-1},[\,])\xleftarrow{\partial_n}(K_n,[\,])\xleftarrow{\partial_{n+1}}(K_{n+1},[\,])\leftarrow\ldots$$

of commutative (4,2)-groups $(K_n,[\,])$, and (4,2)-homomorphisms $\partial_n:(K_n,[\,])\to(K_{n-1},[\,])$, such that for every integer n, $\partial_n\partial_{n+1}=0$, i.e. $\partial_n\partial_{n+1}$ is the zero homomorphism.

7° If w(K,∂) is a weak (4,2)-chain complex, then $B_n = \mathrm{Im}\partial_{n+1}$ and $Z_n = \ker\partial_n$ are (4,2)-subgroups of K_n, and B_n is a (4,2)-subgroup of Z_n, for every integer n. In general, B_n is not a normal (4,2)-subgroup of Z_n.

8° A strong (4,2)-chain complex, denoted by s(K,∂), is a weak (4,2)-chain complex with the additional requirement that B_n is a normal (4,2)-subgroup of Z_n, for every integer n.

9° If w(K,∂) and w(K',∂') are weak (4,2)-chain complexes, then a (4,2)-chain map f from w(K,∂) to w(K',∂') is a sequence of (4,2)-homomorphisms

$$f_n:(K_n,[\,])\to(K'_n,[\,]),\ n - \text{integer}$$

such that $\partial'_n f_n = f_{n-1}\partial_n$, i.e. for every integer n, the following diagram commutes

$$\begin{array}{ccc}(K_{n-1},[\,]) & \xleftarrow{\partial_n} & (K_n,[\,])\\ \downarrow f_{n-1} & & \downarrow f_n\\ (K'_{n-1},[\,]) & \xleftarrow{\partial'_n} & (K'_n,[\,]).\end{array}$$

10° The weak (4,2)-chain complexes and (4,2)-homomorphisms, form a category, denoted by (4,2)-w∂K, whose subcategory is (4,2)-s∂K of the strong (4,2)-chain complexes and (4,2)-homomorphisms.

11° Three functors, denoted by F_2, F_+ and F_* from the category (4,2)-w∂K to the category ∂K of chain complexes of Abelian groups are defined as follows.

For a weak (4,2)-chain complex w(K,∂):

(1) F_2(w(K,∂)) is the sequence of the groups $\Phi_2(K_n)$ with the boundary operators $\Phi_2(\partial_n)$;

(2) F_+(w(K,∂)) is the sequence of the groups $\Phi_+(K_n)$ with the boundary operators $\Phi_+(\partial_n)$; and

(3) F_*(w(K,∂)) is the sequence of the groups $\Phi_*(K_n)$ with the boundary operators $\Phi_*(\partial_n)$.

For a (4,2)-chain map f : w(K,∂) → w(K',∂'):

(1) $F_2(f)$ is the sequence of the homomorphisms $\Phi_2(f_n)$: $\Phi_2(K_n)\to\Phi_2(K_n')$;

(2) $F_+(f)$ is the sequence of the homomorphisms $\Phi_+(f_n)$: $\Phi_+(K_n)\to\Phi_+(K_n')$; and

(3) $F_*(f)$ is the sequence of the homomorphisms $\Phi_*(f_n)$: $\Phi_*(K_n)\to\Phi_*(K_n')$.

12° For any integer n, let H_n: ∂K→*Ab* be the functor such that for a chain complex K=(K,∂), $H_n(K)$ is the n-th homology group of K, and for a chain map f : K → K', $H_n(f) : H_n(K) \to H_n(K')$ is the induced homormphism.

13° For any integer n, by composing the functors F_2, F_+ and F_* with the functor H_n, three functors from (4,2)-w∂K to the category *Ab* are defined as follows.

Let K=w(K,∂), K'=w(K',∂') be two weak (4,2)-chain complexes and let f : K → K' be a chain map. Then:

(1) $H_{n,2}(K) = H_n(F_2(K))$ and $H_{n,2}(f) = H_n(F_2(f))$;

(2) $H_{n,+}(K) = H_n(F_+(K))$ and $H_{n,+}(f) = H_n(F_+(f))$; and

(3) $H_{n,*}(K) = H_n(F_*(K))$ and $H_{n,*}(f) = H_n(F_*(f))$.

Since a strong (4,2)-chain complex K=s(K,∂) is also a weak (4,2)-chain complex, the above homology groups $H_{n2,}(K)$, $H_{n,+}(K)$ and $H_{n,*}(K)$ are defined. Since for a strong (4,2)-chain complex K=s(K,∂), $B_n = \mathrm{Im}\partial_{n+1}$ is a normal (4,2)-subgoup of

$Z_n = \ker\partial_n$, we have the (4,2)-factor group Z_n/B_n.

14^o For any integer n, the functor (4,2)-H_n from the category (4,2)-s∂K to the category (4,2)-*Ab* is defened as follows. For any strong (4,2)-chain complexes K=s(K,∂), (4,2)-H_n(K) is the (4,2)-factor group Z_n/B_n. It is shown in [5], that for a (4,2)-chain map f : K $\to$ K', where K and K' are strong (4,2)-chain complexes, the map (4,2)-H_n(f) defined by (4,2)-$H_n(f)(x^{\sim}) = (f(x))^{\sim}$, for $x\in\ker\partial_n$ is a (4,2)-homomorphism from (4,2)-H_n(K) to (4,2)-H_n(K').

15^o By composing the functors Φ_2, Φ_+ and Φ_* from the category (4,2)-*Ab* to the category *Ab*, with the functor (4,2)-H_n, three functors, $\Phi_2\circ$(4,2)-H_n, $\Phi_+\circ$(4,2)-H_n , and $\Phi_*\circ$(4,2)-H_n , from the category (4,2)-s∂K to the category *Ab* are obtained.

Using the fact that (4,2)-s∂K is a subcategory of (4,2)-w∂K, it is shown in [5], that: $\Phi_2\circ$(4,2)-H_n is the restriction of $H_{n,2}$ on (4,2)-s∂K; $\Phi_+\circ$(4,2)-H_n is the restriction of $H_{n,+}$ on (4,2)-s∂K; and that $\Phi_*\circ$(4,2)-H_n is the restriction of $H_{n,*}$ on (4,2)-s∂K.

(4,2)-CHAIN HOMOTOPY

Let K=w(K,∂) and K'=w(K',∂') be two weak (4,2)-chain complexes, and let f,g : K $\to$ K' be two (4,2)-chain maps.

Let s be a sequence of (4,2)-homomorphisms: $s_n : (K_n, [\ ]) \to (K'_{n+1}, [\ ])$.

The sequence s induces a sequence $\Phi_2(s)$ of homomorhisms

$$(\Phi_2(s))_n = \Phi_2(s_n): ((K_n)^2, \circ) \to ((K'_{n+1})^2, \circ).$$

Definition 1. Let K, K', f, g and s be as above. The sequence s is said to be a **(4,2)-chain homotopy from f to g**, denoted by s: f α g, if for every integer n and any $x,y\in K_n$:

(A) $$[\partial'_{n+1}(s_n(x))\ \partial'_{n+1}(s_n(y))\ s_{n-1}(\partial_n(x))\ s_{n-1}(\partial_n(y))\ g_n(x)\ g_n(y)] = (f_n(x), f_n(y)).$$

Using the operation $\circ$ from $((K'_n)^2,\circ)$, the condition **(A)** can be written in the form

(B) $$\Phi_2(\partial'_{n+1})(\Phi_2(s_n)(x,y))\circ\ \Phi_2(s_{n-1})(\Phi_2(\partial_n)(x,y)) = \Phi_2(f_n)(x,y)\circ(\Phi_2(g_n)(x,y))^{-1}.$$

For every (4,2)-chain map, if for every integer n, we take s_n to be the zero (4,2)-homomorphism, i.e. $s_n(x)=0$, for every x, then, directly from the definition, it follows that s is a (4,2)-chain homotopy from f to f. Hence, the relation α is a reflexive relation.

In general, the relation α is not symmetric, i.e. the existence of a (4,2)-chain homotopy from f to g, does not imply the existence of a (4,2)-chain homotopy from g to f. Also, in general, the relation α is not transitive, i.e. the existence of (4,2)-chain homotopies from f to g and from g to h, does not imply the existence of (4,2)-chain homotopy from f to h.

Although the relation α is not an equivalence relation, it satisfies several properties that will allow us to extend it to an equivalence relation, analogous to the equivalence relation of chain homotopy in the category ∂K of chain complexes of commutative groups.

Next, for (4,2)-chain homotopy s, let: $F_2(s)$ be the sequence defined by $(F_2(s))_n = \Phi_2(s_n)$; $F_+(s)$ be the sequence defined by $(F_+(s))_n = \Phi_+(s_n) = s_n$; and $F_*(s)$ be the sequence defined by $(F_*(s))_n = \Phi_*(s_n)$.

Proposition 1. Let f ,g: K$\to$K' be (4,2)-chain maps and let s be a (4,2)-chain homotopy from f to g, i.e. s: f α g . Then, in the category ∂K, where the chain homotopy is an equivalence relation, $F_2(s)$, $F_+(s)$ and $F_*(s)$ are chain homotopies, i.e. $F_2(s)$: $F_2(f) \sim F_2(g)$; $F_+(s)$: f ~ g; and $F_*(s)$: f ~ g .

Proof. The condition **(B)** implies that $F_2(s)$ is a chain homotopy from $F_2(f)$ to $F_2(g)$. Although, in general, a (4,2)-chain homotopy from g to f, does not exist, the sequence ψ_n: $(K_n)^2\to(K'_{n+1})^2$ defined by $\psi(x,y)=(s_n(x),s_n(y))^{-1}$, is a chain homotopy from $F_2(g)$ to $F_2(f)$. For the transitivity, let s' be a (4,2)-chain homotopy from g to h. Then the sequence Ψ_n: $(K_n)^2\to(K'_{n+1})^2$ defined by

$$\Psi(x,y) = (s_n(x),s_n(y))\circ(s'_n(x),s'_n(y)),$$

is a chain homotopy from $F_2(f)$ to $F_2(h)$.

Next, we look at $F_+(s)$. By setting y=x in **(A)** we obtain $[uuvvg_n(x)g_n(x)] = (f_n(x),f_n(x))$, where

$$u = \partial'_{n+1}(s_n(x)) \text{ and } v = s_{n-1}(\partial_n(x)).$$

This implies that $u + v + g_n(x) = f_n(x)$, i.e.

$$\partial'_{n+1}(s_n(x)) + s_{n-1}(\partial_n(x)) = f_n(x) - g_n(x).$$

Hence, s is a chain homotopy from f to g, i.e. from $F_+(f)$ to $F_+(g)$.

The sequence –s, defined by $(-s)_n(x) = s_n(-x) = -s_n(x)$, is a chain homotopy from $F_+(g)$ to $F_+(f)$.

If s' is a (4,2)-chain homotopy from g to h, the sequence s+s' defined by $(s+s')_n(x)=s_n(x)+s'_n(x)$ is a chain homotopy from $F_+(f)$ to $F_+(h)$.

The discussion for $F_*(s)$ is similar to the discussion for $F_+(s)$. Using the notation for u and v as above, by setting y = –x in **(A)** we obtain $[uu'vv'g_n(x))g_n(-x)] = (f_n(x),f_n(-x))$, where

$$u' = \partial'_{n+1}(s_n(-x)) \text{ and } v' = s_{n-1}(\partial_n(-x)).$$

Since s_n , ∂_n , ∂'_{n+1} and g_n are (4,2)-homomorphisms, it follows that $u' = \partial'_{n+1}(s_n(-x)) = -\partial'_{n+1}(s_n(x)) = -u$,

$v' = s_{n-1}(\partial_n(-x)) = -s_{n-1}(\partial_n(x)) = -v$ and $g_n(-x) = -g(x)$, abd so, $[u(-u)v(-v)g_n(x))g_n(-x)] = (f_n(x),f_n(-x))$. This implies that $u * v * g_n(x) = f_n(x)$, i.e.

$$\partial'_{n+1}(s_n(x)) * s_{n-1}(\partial_n(x)) = f_n(x) * (-g_n(x)).$$

Hence, s is a chain homotopy from f to g, i.e. from $F_*(f)$ to $F_*(g)$.

The sequence $-s$, defined by $(-s)_n(x) = s_n(-x) = -s_n(x)$, is a chain homotopy from $F_*(g)$ to $F_*(f)$.

If s' is a (4,2)-chain homotopy from g to h, the sequence s_*s' defined by $(s_*s')_n(x)=s_n(x)_*s'_n(x)$ is a chain homotopy from $F_*(f)$ to $F_*(h)$. □

Corollary 1. Let f ,g: K→K' be (4,2)-chain maps. A sequence s, of (4,2)-homomorphisms

$$s_n : (K_n, [\ ]) \to (K'_{n+1}, [\ ])$$

is a (4,2)-chain homotopy from f to g, i.e. s: f α g if and only if the sequence $F_2(s)$ of homomorphisms $(F_2(s))_n = \Phi_2(s_n)$ is a chain homotopy from $F_2(f)$ to $F_2(g)$, i.e. $F_2(s)$: $F_2(f) \sim F_2(g)$.

Proof. The proof, follows from Proposition 1 and the condition **(B)**. □

Proposition 2. Let $K=w(K,\partial)$, $K'=w(K',\partial')$ and $K''=w(K'',\partial'')$ be weak (4,2)-chain complexes.

(a) If f , g : K → K' and h : K' → K" are (4,2)-chain maps, and f α g , then hf α hg .

(b) If f : K → K' and g , h : K' → K" are (4,2)-chain maps, and h α g , then hf α gf .

Proof. (a) Let s be a (4,2)-chain homotopy from f to g, i.e. s: f α g . Using the fact that h_n is a (4,2)-homomorphism and applying it to **(A)**, we obtain that

$$[(h_n\partial'_{n+1}s_n(x))(h_n\partial'_{n+1}s_n(y))(h_ns_{n-1}\partial_n(x))(h_n(s_{n-1}(\partial_n(y))(h_ng_n(x))(h_ng_n(y))] = ((h_nf_n(x)),(h_nf_n(y))).$$

Next, using the fact that h is a (4,2)-chain map, i.e. that $h_n\,\partial'_{n+1} = \partial''_{n+1}\,h_{n+1}$, we obtain that

$$[(\partial''_{n+1}(h_{n+1}s_n(x))\,(\partial''_{n+1}(h_{n+1}s_n(y))(h_ns_{n-1}(\partial_n(x))(h_ns_{n-1}(\partial_n(y))\,(h_ng_n(x))(h_ng_n(y))] = ((h_nf_n(x)),(h_nf_n(y))).$$

The last equality is the condition **(A)** for the chain maps hf and hg and for the sequence hs of (4,2)-homomorpisms defined by $(hs)_n = h_{n+1}s_n$. All this implies that hs: hf α hg .

(b) The proof is similar to the prof of **(a)**. Let s be a (4,2)-chain homotopy from g to h. Then the sequence sf of (4,2)-homomorpisms defined by $(fs)_n = f_ns_n$ is a (4,2)-chain homotopy from gf to hf, i.e. fs: gf α hf .□

We denote the symmetric and transititive closure of the relation α (i.e. the smallest equivalence relation containing α) by ~ . With this, the relation ~ is an equivalence relation for the (4,2)-chain maps in the category (4,2)-w∂K, and also in the category (4,2)-s∂K.

Remark 1. The definition of ~ implies that for two (4,2)-chain maps f and g, f ~g if and only if there are (4,2)-chain maps $h_1,h_2,h_3, \ldots,h_m,h_{m+1}$ such that for any $j \in \{1,2,3,\ldots,m\}$, $h_j\ \alpha\ h_{j+1}$ or $h_{j+1}\ \alpha\ h_j$; $f = h_1$; and $g = h_{m+1}$.

Definition 2. Two (4,2)-chain maps f and g are said to be **(4,2)-homotopic** if f ~ g . A (4,2)-chain map f : K → K' is said to be a **(4,2)-homotopy equivalence** if there is a (4,2)-chain map g : K' → K such that $gf \sim 1_K$ and $fg \sim 1_{K'}$ where 1_K and $1_{K'}$ are the identity (4,2)-chain maps for K and K' respectively. Two weak (4,2)-chain complexes $K=w(K,\partial)$, $K'=w(K',\partial')$ are said to be **(4,2)-homotopy equivalent,** denoted by K ~ K', if there is a (4,2)-homotopy equivalence f : K → K' . Two strong (4,2)-chain complexes are said to be **(4,2)-homotopy equivalent,** if they are (4,2)-homotopy equivalent as weak (4,2)-chain complexes.

Proposition 3. Let K, K', K" be (4,2)-chain complexes, and let g,h: K→ K', g',h':K'→ K" be (4,2)-chain maps, such that g ~ h and g' ~ h'. Then, the compositions gg', hh': K→ K" are (4,2)-homotopic, i.e. gg'~ hh'.

Proof. Proposition 2 (a) and Remark 1 imply that g'g ~ g'h, and Proposition 2 (b) and Remark 1 imply that g'h ~ h'h. Thus, gg'~ hh'. □

For a (4,2)-chain map h, we denote the equivalence class $h^{\sim} = \{g \mid g \sim h\}$, by [h].

Proposition 3 implies the following:

Corollary 2. All $w(K,\partial)$, the weak (4,2)-chain complexes as objects and all [h], the (4,2)-homotopy classes of (4,2)-chain maps, form a category, denoted by (4,2)-hw∂K. All strong (4,2)-chain complexes and all (4,2)-homotopy classes of (4,2)-chain maps, form a subcategory of (4,2)-hw∂K, denoted by (4,2)-hs∂K.

Proposition 4. If h,g are two (4,2)-homo-topic (4,2)-chain maps, i.e. h ~ g, then their images by the functors F_2, F_+ and F_* in the category of chain complexes ∂K are homotopic maps, i.e. $F_2(h) \sim F_2(g)$, $F_+(h) \sim F_+(g)$ and $F_*(h) \sim F_*(g)$.

Proof. The proof follows from Remark 1 and Proposition 1. □

Proposition 4 implies the following:

Corollary 3. The functors F_2, F_+ and F_* produce three functors, denoted by the same notation, from the category (4,2)-hw∂K to the category h∂K, whose objects are the chain complexes of commutative groups, and the morphisms are the homotopy classes of chain maps.

Proposition 5. If h,g are two (4,2)-homotopic (4,2)-chain maps, then their images by the homology functors $H_{n,2}$, $H_{n,+}$ and $H_{n,*}$ are equal, i.e. $H_{n,2}(h)=H_{n,2}(g)$, $H_{n,+}(h)=H_{n,+}(g)$ and $H_{n,*}(h)=H_{n,*}(g)$.

Proof. The proof follows from Proposition 4 together with the fact that homotopic chain maps in the category ∂K have the same images by the homology functors H_n. □

Proposition 6. Let K=w(K,∂), K'=w(K',∂') be two strong (4,2)-chain complexes and let h,g: K$\to$K' be (4,2)-homotopic (4,2)-chain maps. Then the (4,2)-homomorphisms (4,2)-H_n(h) and (4,2)-H_n(g) are equal.

Proof. Since (4,2)-$H_n(K) = \ker\partial_n/\mathrm{Im}\partial_{n+1}$ and (4,2)-$H_n(K')=\ker\partial'_n/\mathrm{Im}\partial'_{n+1}$, it is sufficient to show that for any $x\in\ker\partial_n$, $h_n(x) \sim g_n(x)$, where ~ is the equivalence relation defined in **4°**, for G = $\ker\partial'_n$ and H = $\mathrm{im}\partial'_{n+1}$, i.e. it is sufficient to show that:

$$h_n(x) - g_n(x)\in \mathrm{im}\partial'_{n+1}, \text{ for any } x\in \ker\partial_n .$$

By Remark 1, it is sufficient to consider the case when h α g. Let s be a (4,2)-chain homotopy from h to g. Then, Proposition 1 implies that $F_+(h)$ and $F_+(g)$ are chain homotopic, i.e. that

$$\partial'_{n+1}(s_n(x)) + s_{n-1}(\partial_n(x)) = h_n(x) - g_n(x),$$

for every integer n and any $x\in K_n$. This, together with the fact that $\partial_n(x) = 0$ for $x\in\ker\partial_n$ implies that

$$h_n(x) - g_n(x) = \partial'_{n+1}(s_n(x)) + 0 = \partial'_{n+1}(s_n(x)),$$

i.e. that $h_n(x) - g_n(x) \in \mathrm{im}\partial'_{n+1}$.□

As a consequence of Propositions 5 an 6, we obtain the following corollaries.

Corollary 4. (a) If h is a (4,2)-homotopy equivalence in (4,2)-w∂K, then $H_{n,2}(h)$, $H_{n,+}(h)$ and $H_{n,*}(h)$ are isomorphisms.

(b) If h is a (4,2)-homotopy equivalence in (4,2)-s∂K, then (4,2)-H_n(h) is a (4,2)-isomorphism.

Corollary 5. If K and K' are (4,2)-homotopy equivalent (4,2)-chain complexes, then:

(1) $H_{n,2}(K)$ and $H_{n,2}(K')$ are isomorphic groups;
(2) $H_{n,+}(K)$ and $H_{n,+}(K')$ are isomorphic groups; and
(3) $H_{n,*}(K)$ and $H_{n,*}(K')$ are isomorphic groups.

Moreover, if K and K' are strong (4,2)-chain complexes, then (4,2)-H_n(K) and (4,2)-H_n(K') are isomorphic (4,2)-commutative groups.

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(4,2)-ВЕРИЖНА ХОМОТОПИЈА

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Разгледувана е (4,2)-верижна хомотопија за (4,2)-верижни пресликувања помеѓу (4,2)-верижни комплекси (слаби или јаки) и докажано е дека ако f и g се (4,2)-верижно хомотопни (4,2)-верижни пресликувања, тие индуцираат исти хомоморфизми на (4,2)-хомолошките групи од соодветните (4,2)-верижни комплекси.

Клучни зборови: комутативни (4,2)-групи; слаб (4,2)-верижен комплекс; јак (4,2)-верижен комплекс; (4,2)-верижно пресликување; (4,2)-верижна хомотопија